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Nonlinear Quantum Mechanics at the Planck Scale

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I argue that the linearity of quantum mechanics is an emergent feature at the Planck scale, along with the manifold structure of space-time. In this regime the usual causality violation objections to nonlinearity do not apply, and nonlinear effects can be of comparable magnitude to the linear ones and still be highly suppressed at low energies. This can offer alternative approaches to quantum gravity and to the evolution of the early universe.

KEY WORDS: nonlinear quantum mechanics; Planck-scale physics; quantum gravity.

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1. INTRODUCTION

I adopt what would be a physicist's conception of nonlinear quantum mechanics (NLQM): a theory that (1) at low energies reduces to standard linear quantum mechanics, and (2) involves, in an essential way, nonlinear operators in lieu of linear ones. This is not the view adopted in a large part of the literature dedicated to the subject. Studies of abstract structures may ignore the first point if no immediate confrontation with reality is contemplated, and the second point may only be implicit. At a meeting such as this IQSA 2004, anyone who considers quantum logics not representable in Hilbert is probably dealing with NLQM. For my purposes, the stated view is essential. I should quickly point out though that there is no such theory, at best one has only a few exploratory results.

It may be useful to begin with some history and try to answer the *who?*, *what?*, *how*?, and *why*? of the field. NLQM has a small literature with very varied content. A rough survey in arXiv reveals roughly 121 articles by 97 authors since 1991. I posted a list of these on the archives (Svetlichny, 2004c).

The above survey of course leaves out contributions from the pre-Internet era and some important references such as Bialynicki-Birula and Mycielski (1976), Bugajski (1991), Czachor (1991), Gisin (1984, 1989, 1990), Haag and Bannier (1978), Kibble (1979), Kostin (1972), Polchinski (1991), and Weinberg (1989a,b).

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From this one can guess that the total literature contains at most a few hundred works by roughly the same number of authors. Thus few authors contribute regularly, most contribute once or sporadically. This is already an unusual situation for a topic that has been around for over 20 years.

What is also unusual are some of the things *said* about NLQM, for instance: NLQM (1) is essentially classical: Bugajski (1991), Haag and Bannier (1978); (2) violates causality: Czachor (1991), Polchinski (1991), Gisin (1984, 1989, 1990), Svetlichny (1998), Lücke (1999); (3) allows communication between Everett histories or makes experiments react to the content of the experimenter's mind: Polchinski (1991); (4) is necessary in the presence of closed time-like curves: Cassidy (1995); (5) may violate space-time symmetries: Parwani (2004), Svetlichny (1995); (6) is necessary for homogeneous quantum cellular automata on Euclidean space: Meyer (1996); (7) solves NP-complete and #P problems in polynomial time: Abrams and Lloyd (1998); (8) is necessary for introspection: Hübsch (1998); (9) solves the "measurement problem": Hansson (2000); (10) is involved in black-hole dynamics: Yurtserver and Hockney (2005).

Now I shall not discuss the merit of any of these claims, but one can wonder why NLQM forces us to consider and reevaluate such a great variety of some very fundamental scientific issues. My speculative answer to this is that the observed quantum linearity is related to space-time structure, and space-time obviously bears upon all our fundamental concerns. Space-time and quantum mechanics would thus be a unified whole, emergent aspects of a more fundamental theory, and one cannot understand the one without the other. At the emergence level, quantum mechanics may very well be nonlinear and linearity comes about because it must eventually act in a space-time arena.

2. WAYS TOWARD NLQM

There are two ways leading to NLQM.

Willing. Here nonlinearity is posited from the beginning, due to intelectual speculation or axiomatics. Most proposals fall into this category.

Unwilling. One is surprisingly forced to consider nonlinearities in ostensively linear contexts. Examples are: (1) representations of current algebras: Doebner and Goldin (1992); (2) quantum evolution in acausal space-times: Cassidy (1995); (3) quantum cellular automata on Euclidean lattices: Meyer (1996); (4) introspective quantum mechanics: Hübsch (1998); (5) dynamics of D0 branes in noncritical string theory: Mavromatos and Szabo (2001).

The degree of "surprise" is of course subjective and my personal "unwilling" list is unstable, but the first item seems to be always firmly in place.

One of the motivations for the willing is solution of some fundamental problem of contemporary science, in particular in quantum gravity, cosmology, quantum mechanics, computation, and cognition. Now since NLQM calls for a broadband modification of all our fundamental physical theories, one can expect that *any* nonlinear quantum mechanical theory will *appear* to solve some fundamental problems, that is, address them better than the existing theories. This means that a resolution of one or other of these problems by a nonlinear theory cannot be considered a strong reason for its adoption.

Based on the above considerations I have adopted the following guiding rules for trying to approach the hypothetical true nonlinear theory. (1) The unwilling nonlinearities are more likely to be closer to the true theory than the willing ones. (2) Widespread properties of studied nonlinearities are more likely to be true of the true theory. (3) One should not be motivated by the desire to solve any particular "fundamental problem."

Based on this I here will focus on the (unwilling) Doebner–Goldin nonlinearities (Doebner and Goldin, 1992) and address the (widespread) causality issue (Czachor, 1991; Gisin, 1984, 1989, 1990; Luecke, 1991; Polchinski, 1991; Svetlichny, 1998).

3. NONLINEAR SCHRÖDINGER EQUATIONS

Doebner and Goldin (1992) studied representations of nonrelativistic current algebras, which in particular involve unitary representations of the diffeomorphism group of ordinary Euclidian space $\mathbb{R}^n$. From such a representation one can construct the density ρ and current $\mathbf{J}$ operators of a nonrelativistic quantum theory. For one such notable representation these operators do not satisfy a continuity equation but instead a Fokker–Planck equation: $\partial_t \rho = -\nabla \cdot \mathbf{J} + D\nabla^2 \rho$ where D is a physical constant. No linear quantum system is consistent with this, but nonlinear ones are, the simplest given by the Doebner–Goldin equation

$$i\hbar\partial_t \psi = -\frac{\hbar^2}{2m}\Delta\psi + iD\hbar\left(\Delta\psi + \frac{|\nabla\psi|^2}{|\psi|^2}\psi\right). \tag{1}$$

One can add to the right-hand side a term $R(\psi)\psi$ where $R(\psi)$ is *real* obeying $R(z\psi) = R(\psi)$ for complex z.

Now representations of the diffeomorphism group is certainly a highly respectable mathematical topic. That nonlinear quantum systems are somehow connected to them is probably the strongest reason to give them further thought, especially since diffeomorphism related issues are germane to physics at the Planck level.

One important property that a nonlinear evolution can satisfy is *separability*, a nonlinear generalization of lack of interaction and which states that noncorrelated systems continue noncorrelated. Assume the evolution is governed by a not necessarily linear Schrödinger equation $i\hbar\partial_t \Psi_s = H_s\Psi_s$ where Ψ_s is an N-particle wave function and $s = (s_1, \ldots, s_N)$ indicates the species of each particle. All particles belong to different species. One necessarily has (Goldin and Svetlichny, 1994):

$$H_s\Psi = K_s\Psi + p\ln|\Psi_s|\,\Psi_s + iq\ln(\Psi_s/\bar{\Psi}_s)\,\Psi_s$$

where $p\ln|\Psi_s|\,\Psi_s$ is the Bialynicki–Birula and Mycielski (1976) term, $iq\ln(\Psi_s/\bar{\Psi}_s)\,\Psi_s$ is the Kostin (1972) term, p and q are universal physical constants, and K_s is homogeneous $K_s(z\Psi) = zK_s(\Psi)$.

A two-particle Schrödinger operator is built-up from one particle operators by

$$K_{ab}\Psi = K_a^{(1)}\Psi + K_b^{(2)}\Psi + Q\Psi \tag{2}$$

where $K_s^{(i)}$ is a one-particle operator acting on the i-th variable of Ψ and Q is an operator that vanishes identically on product functions. This generalizes to an n-particle operator construction and one can introduce true n-particle effects that don't exist for smaller number of particles.

The case for identical particles is more subtle. There are no nonlinear separating hierarchies of differential Schrödinger operators (Svetlichny, 2004b) and this constitutes another indication that linearity has something to do with space-time structure.

4. (NON)LINEARITY, SPACE-TIME, AND CAUSALITY

Many of the claims about NLQM mentioned in the Introduction point to a connection with space-time. In Svetlichny (2000), I present a deduction of the covering law, a close relative of linearity, through a local relativistic quantum logic, specifically: (1) Lorentz covariance; (2) causality, that is, propositions belonging to space-like separated regions commute; (3) state-collapse; and (4) an abundance of space-like separated entangled states to be able to "prepare at a distance" any given state (true of local relativistic quantum mechanics).

The first interesting fact about this is that if one insists on eliminating nonlocal state-collapse, one cannot complete the deduction. This makes quantum mechanics understandable only if one combines relativity and causality with some form of nonlocality. Such nonlocality would be natural in a quantum space-time. The second fact is that if the argument shows universality of quantum mechanics, it must also apply to space-time coordinate measurements. However, the argument assumes a classical Minkowski space-time and so in the end is contradictory. The conclusion is inescapable: *only* quantum *space-time can make quantum mechanics intelligible*.

Hence I come to my main conjecture, which was also voiced by other authors: *Linear quantum mechanics is an emergent feature of "quantum gravity" which may very well be nonlinear*.

See Markopoulou and Smolin (2003), Parwani (2004), Singh (2003), and Svetlichny (2004a,b).

The appearance of superluminal signals seems to be a generic feature of nonlinear quantum theories. There have been many proposals for circumventing this apparent violation of causality, but one still cannot say that we have an explicit and consistent causal relativistic nonlinear quantum mechanical theory. However, *if nonlinearities are of Plank scale should one worry*?

At Planck energies space-time is thought to be ill defined, the causal structure also ill defined, and so it makes little sense to talk of its violation. Lorentz invariance itself may be broken (Amelino-Camelia, 2003), a hypothesis put forth to explain some cosmic ray phenomena (see Section 5), which further casts doubt on the ultimate seriousness of the causality violation issues. In the end all the space-time difficulties of NLQM may not be pernicious. It would take Planck energies to exhibit the effects, but then space-time itself becomes quantum and the apparent problems could have no problematic low energy consequences. The presence of such effects at the Planck scale could, however, completely transform our understanding of quantum space-time.

5. EXPERIMENTAL SITUATION

A series of experiments designed to test nonlinear effects of the Weinberg (1989a,b) type show that these are about 10^{-20} times smaller than linear ones (Benatti and Floreanini, 1996, 1999; Bollinger *et al.*, 1989; Chupp and Hoare, 1990; Majumder *et al.*, 1990; Walsworth *et al.*, 1990). While this is consistent with the hypothesis that such putative effects would only appear on the Planck scale, one is inevitably led to ask, if so, how can one become aware of them? Now there is at least one (possible) physical phenomenon for which NLQM is a ready-made explanation. This occurs in cosmic ray physics (Svetlichny, 2004a and references therein).

Cosmic rays can scatter off the cosmic microwave background, with the cross section increasing with energy. As there are no known nearby sources of such rays, one should not see any above a certain energy (the so called GZK cutoff). Apparently about 20 such events have been seen, and though the existence of this effect is still being debated, speculations abound concerning new physics that would explain them, such as quantum gravity, noncommutative space-time, Lorentz symmetry breaking, etc.

All such explanations propose a modified dispersion relation instead of the usual $E^2 = m^2c^4 + p^2$, typically:

$$E^2 = m^2c^4 + p^2 + \kappa \ell_p p^3 \tag{3}$$

where κ is of order unity and ℓ_p the Planck length $(\hbar G/c^3)^{1/2} \approx 10^{-33}$cm. The highest cosmic ray energy seen is about $10^{20.5}$ eV, whose de Broglie wavelength is then about 3.17×10^7 Planck lengths. Thirty million may seem large, but it is small enough that deviations from a smooth manifold structure can already influence the propagation of the particle. Such particles provide us with true quantum gravity experiments.

If one did not already have some beginnings of quantum gravity and non-commutative space-time theories, the reaction to being forced to use (3) could be: (1) Lorentz covariance is broken, (2) one must use higher order differential equations, or (3) quantum mechanics is nonlinear, which in my view is the simplest *prima-facie* explanation.

6. PLANCK-SCALE NONLINEAR QUANTUM EFFECTS

If nonlinear quantum effects exist at Planck energies, how large can they be and still be consistent with the large experimental suppression at low energies? This is a hard question to get a handle on given a lack of high-energy nonlinear theories but one may get a hint looking at nonlocal signaling due to separated measurements. Now at the Planck scale, say in the early universe, there are no observers measuring things, however there should be decoherence effects having similar consequences (Hansson, 2000). This makes measurement-related arguments relevant.

Consider a not necessarily linear norm-preserving evolution described by a Schrödinger-type equation: $i\hbar\partial_t\Psi = H\Psi$. Consider three conventional quantum observables A, A', and B with $[A, B] = [A', B] = 0$. Let $\mathcal{E}(B, t|A)$ be the expected value of B in the mixture resulting from a measurement of A on a state Φ followed by evolution for time t, and let $\Delta(B, t|A, A') = \mathcal{E}(B, t|A) - \mathcal{E}(B, t|A')$. This quantity vanishes for linear evolution but for a nonlinear one may not, and quantifies the prototypical causality violating effect (assuming B space-like to A and A'). We have the Taylor series $\Delta(B, t|A, A') = t\Delta_1(B|A, A') + O(t^2)$.

Consider now the Doebner–Goldin equation (1) with the two- particle equation as in (2). The initial state Φ is one of zero total momentum and one performs either a momentum ($A = p$) or a position ($A' = q$) measurement on the first particle. One finds after some analysis (Svetlichny, 2004d):

$$\Delta_1(B|p, q) = 2D_b \int \mathrm{Re}\,(B\delta_w, (\Delta + N)\delta_w)\, d\mu(w)$$

where $\delta_w(y) = \delta(y - w)$, $N\psi = (|\nabla\psi|^2/|\psi|^2)\psi$, μ a measure, and D_b the coefficient of the nonlinear term of the second particle.

Now N is ill defined on δ so one uses a gaussian regularization $\delta^{(s)}(y) = (s/\pi)^{n/2}e^{-sy^2}$ with n the dimension of space. As $s \to \infty$ one has, as distributions, $\delta^{(s)}(y) = \delta(y) + O(s^{-1})$.

Asymptotic analysis now shows (Svetlichny, 2004d):

$$\Delta_1(B|p, q) = 4snD_b(\phi, B\phi) + O(1).$$

Thus *even if the physical constant D_b is extremely small, under extreme localization ($s \to \infty$), the effect can be large.*

It is probably significant that not all nonlinear terms have this amplification effect, which seems to be a property of the precise diffeomorphism motivated nonlinearity.

7. SUMMARY

Various results in the literature suggest that linearity of quantum mechanics is an emergent feature of physical processes that must take place in a classical space-time. At the Planck scale nonlinear effects may be present and *may be of the same order of magnitude as linear ones and still suffer large suppression at low energies*. If such nonlinearities exist they would significantly alter our theories of physics at the Plank scale and can offer a new alternative to current Planck-scale physics such as loop quantum gravity, M-theory, brane-world scenarios, quantum cosmology, etc.

To conclude I wish to present one final consideration that may make nonlinear quantum gravity plausible. Consider the familiar general relativistic dictum (apparently to to J. A. Wheeler):

Matter tells space-time how to curve, space-time tells matter how to move.

Let me put a quantum "spin" on this:

Quantum matter tells quantum space-time how to be, quantum space-time tells quantum matter how to behave.

This is not a "final" view, just the next step down from the present quantum-mechanics/general-relativity confrontation, a dichotomy still exists (space and matter), and it's up to better insights to go deeper.

Now quantum matter moves, in a first approximation, by a hamiltonian. By the quantum dictum, the hamiltonian must now depend on quantum matter, this turns the quantum process nonlinear, as there is a back-reaction of matter on its own dynamics.

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